8. Turing Machine

Turing formulated a model of algorithm or computation, that is widely accepted. The Church-Turing thesis states that any algorithmic procedure that can be carried out by human beings/computer can be carried out by a Turing machine. It has been universally accepted by computer scientists that the Turing machine provides an ideal theoretical model of a computer.

For formalizing computability, Turing assumed that, while computing, a person writes symbols on a one-dimensional tape which is divided into cells. One scans the cells one at a time and usually performs one of the three simple operations, namely

(i) writing a new symbol in the cell being currently scanned,

(ii) moving to the cell left of the present cell and

(iii) moving to the cell light of the present cell. With these observations in mind, Turing proposed his 'computing machine.'

8.1 Turing Machine Model

The Turing machine can be thought of as finite control connected to a R/W (read/write) head. It has one tape which is divided into a number of cells. The block diagram of the basic model for the Turing machine is given below.

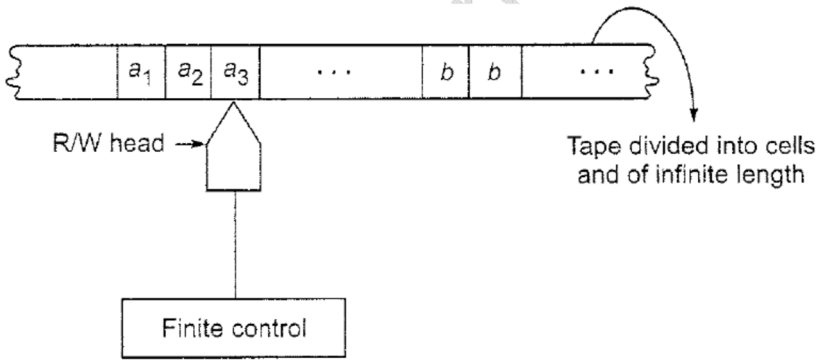


Fig. 9.1 Turing machine model.

Each cell can store only one symbol. The input to and the output from the finite state automaton are affected by the R/W head which can examine one cell at a time. In one move, the machine examines the present symbol under the R/W head on the tape and the present state of an automaton to determine

(i) a new symbol to be written on the tape in the cell under the R/W head,

(ii) a motion of the R/W head along the tape: either the head moves one cell left (L). or one cell right (R),

(iii) the next state of the automaton, and

(iv) whether to halt or not.

The above model can be rigorously defined as follows.

Definition: A Turing machine M is a 7-tuple, namely **(Q, Σ, Γ, δ, q_0, b, F),** where

- Q is a finite nonempty set of states.
- Γ is a finite nonempty set of tape symbols,
- b is the blank.
- Σis a nonempty set of input symbols and is a subset of Γ rand b ∉ Σ.
- δ is the transition function mapping (q, x) onto (q', y, D) where D denotes the direction of movement of R/W head D = L or R according as the movement is to the left or right.
- $q_0 \in Q$ is the initial state, and
- $F \subseteq Q$ is the set of final states.

8.2. Representation of Turing Machines

We can describe a Turing machine employing

(i) Transition diagram (transition graph).
(ii) Instantaneous descriptions using move-relations.
(iii) Transition table

8.2.1. Representation by Instantaneous Descriptions

'Snapshots' of a Turing machine in action can be used to describe a Turing machine. These give 'instantaneous descriptions' of a Turing machine. An ID of a Turing machine is defined in terms of the entire input string and the current state.

Definition: An ID of a Turing machine M is a string $\alpha\beta\gamma$, where β is the present state of M, the entire input string is split as $\alpha\gamma$, the first symbol of γ is the current symbol a under the R/W head and γ has all the subsequent symbols of the input string, and the string α is the substring of the input string formed by all the symbols to the left of a.

Example: A snapshot of Turing machine is shown in. Obtain the instantaneous description.

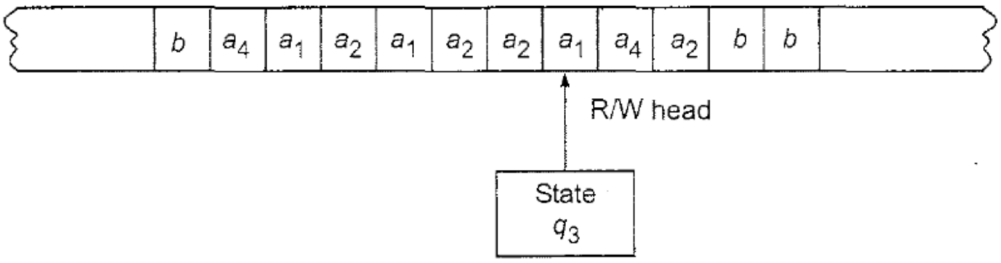


Fig. 9.2 A snapshot of Turing machine.

Solution: The present symbol under the R/W head is a_1. the present state is q_3 So a_1 is written to the right of q_3. The nonblank symbols to the left of al form the string $a_4a_1a_2a_1a_2a_2$, which is written to the left of q_3. The sequence of nonblank symbols to the right of a_1 is a_4a_2. Thus, the ID is as given in figure given below.

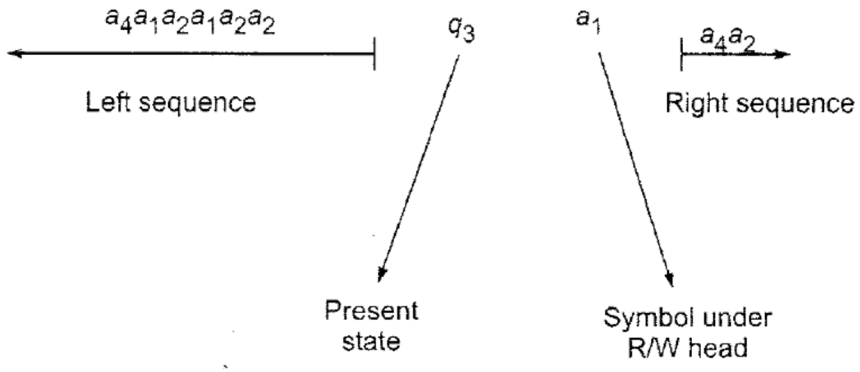


Fig. 9.3 Representation of ID.

Note: (1) For constructing the ID, we simply insert the current state in the input string to the left of the symbol under the R/W head. (2) We observe that the blank symbol may occur as part of the left or right substring.

As in the case of pushdown automata, $\delta(q, x)$ induces a change in ID of the Turing machine. We call this change in ID a move.

Suppose $\delta(q, x_i) = (p, y, L)$. The input string to be processed is $x_1x_2 \ldots x_n$, and the present symbol under the R/W head is x_i. So the ID before processing x_i is

$$x_1x_2 \ldots x_{i-1}qx_i \ldots x_n$$

After processing x_i, the resulting ID is

$$x_1 \ldots x_{i-2}\, px_{i-1}yx_{i+1} \ldots x_n$$

This change of ID is represented by

$$x_1x_2 \ldots x_{i-1}\, q\, x_i \ldots x_n \vdash x_i \ldots x_{i-2}\, px_{i-1}\, yx_{i+1} \ldots x_n$$

If $i = 1$, the resulting ID is $p\, y\, x_2x_3 \ldots x_n$.

If $\delta(q, x_i) = (p, y, R)$, then the change of ID is represented by

$$x_1x_2 \ldots x_{i-1}q\, x_i \ldots x_n \vdash x_1x_2 \ldots x_{i-1}\, y\, px_{i+1} \ldots x_n$$

If $i = n$, the resulting ID is $x_1x_2 \ldots x_{n-1}\, y\, p\, b$.

We can denote an ID by I_j for some j. $I_j \vdash I_k$ defines a relation among IDs. So the symbol $\vdash^*$ denotes the reflexive–transitive closure of the relation $\vdash$. In particular, $I_j \vdash^* I_j$. Also, if $I_1 \vdash^* I_n$, then we can split this as $I_1 \vdash I_2 \vdash I_3 \vdash \ldots \vdash I_n$ for some IDs, $I_2, \ldots, I_{n-1}$.

Note: The description of moves by IDs is very much useful to represent the processing of input strings.

8.2.2. Representation by Transition Table

We give the definition of δ in the form of a table called the transition table. If $\delta(q, a) = (\gamma, \alpha, \beta)$, we write αβγ under the α-column and in the *q-row*. So if we get αβγ in the table, it means that α is written in the current cell, β gives the movement of the head (L or R) and γ denotes the new state into which the Turing machine enters.

Consider, for example, a Turing machine with five states $q_1, \ldots, q_5$, where q_1 is the initial state and q_5 is the (only) final state. The tape symbols are 0, 1 and b. The transition table given in Table 9.1 describes δ.

TABLE 9.1 Transition Table of a Turing Machine

Present state	Tape symbol		
	b	0	1
$\rightarrow q_1$	$1Lq_2$	$0Rq_1$	
q_2	bRq_3	$0Lq_2$	$1Lq_2$
q_3		bRq_4	bRq_5
q_4	$0Rq_5$	$0Rq_4$	$1Rq_4$
(q_5)	$0Lq_2$		

8.2.3. Representation by Transition Diagram

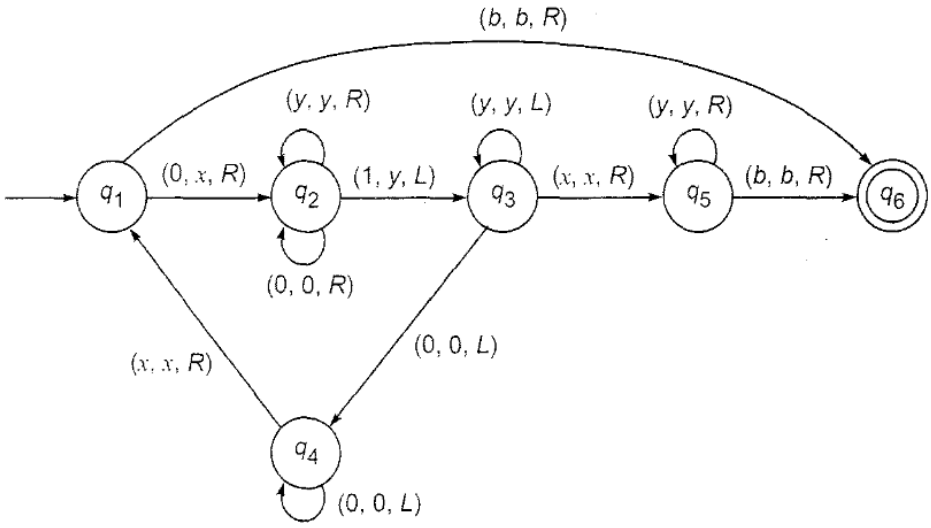
The states are represented by vertices. Directed edges are used to represent transition of states. The labels are triples of the form (α, β, γ), where $\alpha, \beta \in \Gamma$ and $\gamma \in \{L, R\}$. When there is a directed edge from q_i to q_j with label (α, β, γ), it means that

$$\delta(q_i, \alpha) = (q_j, \beta, \gamma)$$

During the processing of an input string, suppose the Turing machine enters q_i and the R/W head scans the (present) symbol α. As a result, the symbol β is written in the cell under the R/W head. The R/W head moves to the left or to the right, depending on γ, and the new state is q_j.

Every edge in the transition system can be represented by a 5-tuple $(q_i, \alpha, \beta, \gamma, q_j)$. So each Turing machine can be described by the sequence of 5-tuples representing all the directed edges. The initial state is indicated by $\rightarrow$ and any final state is marked with $\bigcirc$.

Example:



8.3. Language acceptability by Turing Machines

Let us consider the Turing machine $M = (Q, \Sigma, \Gamma, \delta, q_0, b, F)$. A string w in Σ^* is said to be accepted by M if $q_0w \vdash^* \alpha_1 p \alpha_2$ for some $p \in F$ and $\alpha_1, \alpha_2 \in \Gamma^*$.

M does not accept w if the machine M either halts in a nonaccepting state or does not halt.

Example: Consider the Turing machine M described by the transition table given in Table. Describe the processing of (a) 011 (b) 0011, (c) 001 using IDs. Which of the above strings are accepted by M?

Present state	Tape symbol				
	0	1	x	y	b
$\rightarrow q_1$	xRq_2				bRq_5
q_2	$0Rq_2$	yLq_3		yRq_2	
q_3	$0Lq_4$		xRq_5	yLq_3	
q_4	$0Lq_4$		xRq_1		
q_5				$yxRq_5$	bRq_6
(q_6)					

Solution:

(a) $q_1011 \vdash xq_211 \vdash q_3xy1 \vdash xq_5y1 \vdash xyq_51$

As $\delta(q_5, 1)$ is not defined, M halts; so the input string 011 is not accepted.

(b) $q_10011 \vdash xq_2011 \vdash x0q_211 \vdash xq_30y1 \vdash q_4x0y1 \vdash xq_10y1.$

$\vdash xxq_2y1 \vdash xxyq_21 \vdash xxq_3yy \vdash xq_3xyy \vdash xxq_5yy$

$\vdash xxyq_5y \vdash xxyyq_5b \vdash xxyybq_6$

M halts. As q_6 is an accepting state, the input string 0011 is accepted by M.

(c) $q_1001 \vdash xq_201 \vdash x0q_21 \vdash xq_30y \vdash q_4x0y$

$\vdash xq_10y \vdash xxq_2y \vdash xxyq_2$

M halts. As q_2 is not an accepting state, 001 is not accepted by M.

8.4 Design of Turing Machines

The basic guidelines for designing a Turing machine is given below.

1. The fundamental objective in scanning a symbol by the R/W head is to 'know' what to do in the future. The machine must remember the past symbols scanned. The Turing machine can remember this by going to the next unique state.
2. The number of states must be minimized. This can be achieved by changing the states only when there is a change in the written symbol or when there is a change in the movement of the R/W head.

We shall explain the design by a simple example.

Example-1:

Design a Turing machine to recognize all strings consisting of an even number of 1's.

Solution:

The construction is made by defining moves in the following manner:

(a) q_1 is the initial state. M enters the state q_2 on scanning 1 and writes b.
(b) If M is in state q_2 and scans 1, it enters q_1 and writes b.
(c) q_1 is the only accepting state.

So M accepts a string if it exhausts all the input symbols and finally is in state q_1. Symbolically,

$$M = (\{q_1, q_2\}, \{1, b\}, \{1, b\}, \delta, q, b, \{q_1\})$$

where δ is defined by Table 9.3.

Present state	1
$\rightarrow(q_1)$	bq_2R
q_2	bq_1R

Transition diagram

Let us obtain the computation sequence of 11. Thus, $q_1 11 \vdash bq_2 1 \vdash bbq_1$. As q_1 is an accepting state, 11 is accepted. $q_1 111 \vdash bq_2 11 \vdash bbq_1 1 \vdash bbbq_2$. M halts and as q_2 is not an accepting state, 111 is not accepted by M.

Example-2:

Design a Turing machine over $\{1, b\}$ which can compute a concatenation function over $\Sigma = \{1\}$. If a pair of words (w_1, w_2) is the input, the output has to be $w_1 w_2$.

Solution

Let us assume that the two words, w_1 and w_2 are written initially on the input tape separated by the symbol b. For example, if $w_1 = 11$, $w_2 = 111$. then the input and output tapes are as shown in Figure.

Fig. 9.6 Input and output tapes.

Let us assume that the two words w_1 and w_2 are written initially on the input tape separated by the symbol b. For example, if $w_1 = 11$, $w_2 = 111$, then the input and output tapes are as shown in Fig. 9.6.

Fig. 9.6 Input and output tapes.

We observe that the main task is to remove the symbol b. This can be done in the following manner:

1. The separating symbol b is found and replaced by 1.
2. The rightmost 1 is found and replaced by a blank b.
3. The R/W head returns to the starting position.

We can construct the transition table as follows

Present state	*Tape symbol*	
	1	b
$\rightarrow q_0$	$1Rq_0$	$1Rq_1$
q_1	$1Rq_1$	bLq_2
q_2	bLq_3	—
q_3	$1Lq_3$	bRq_f
(q_f)	—	—

The transition diagram is given here.

A computation for **11b111** is illustrated below.

$$q_0 11b111 \vdash 1q_0 1b111 \vdash 11q_0 b111 \vdash 111q_1 111$$
$$\vdash 1111q_1 11 \vdash 11111q_1 1 \vdash 111111q_1 b \vdash 11111q_2 1b$$
$$\vdash 1111q_3 1bb \vdash 111q_3 11bb \vdash 11q_3 111bb \vdash 1q_3 1111bb$$
$$\vdash q_3 11111bb \vdash q_3 b11111bb \vdash bq_f 11111bb$$

For the input string $1b1$, the computation sequence is given as

$$q_0 1b1 \vdash 1q_0 b1 \vdash 11q_1 1 \vdash 111q_1 b \vdash 11q_2 b \vdash 1q_3 1bb$$
$$\vdash q_3 11bb \vdash q_3 b11bb \vdash bq_f 11bb.$$

Example-3: Design TM that accepts $\{0^n1^n \mid n \geq 1\}$

Solution:

We require the following moves:

(a) If the leftmost symbol in the given input string w is 0, replace it by x and move right till we encounter a leftmost 1 in w. Change it to y and move backwards.

(b) Repeat (a) with the leftmost 0. If we move back and forth and no 0 or 1 remains, move to a final state.

(c) For strings not in the form 0^n1^n, the resulting state has to be nonfinal.

Keeping these ideas in our mind, we construct a TM M as follows:

where

$$M = (Q, \Sigma, \Gamma, \delta, q_0, b, F)$$

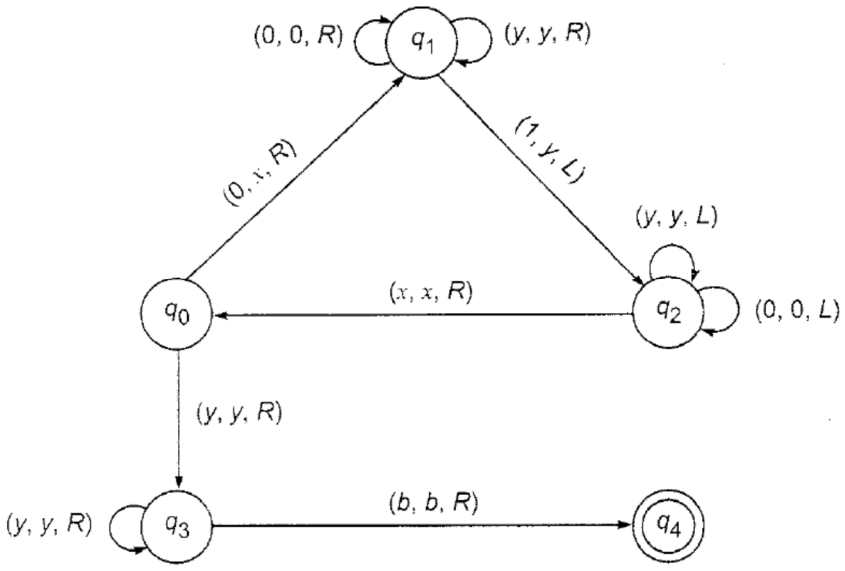
$$Q = \{q_0, q_1, q_2, q_3, q_f\}$$

$$F = \{q_f\}$$

$$\Sigma = \{0, 1\}$$

$$\Gamma = \{0, 1, x, y, b\}$$

The transition diagram is given in Fig. 9.7. M accepts $\{0^n1^n \mid n \geq 1\}$.



The moves for 0011 and 010 are given below.

$$q_00011 \vdash xq_1011 \vdash x0q_111 \vdash xq_20y1$$
$$\vdash q_2x0y1 \vdash xq_00y1 \vdash xxq_1y1 \vdash xxyq_11$$
$$\vdash xxq_2yy \vdash xq_2xyy \vdash xxq_0yy \vdash xxyq_3y$$
$$\vdash xxyyq_3 = xxyyq_3b \vdash xxyybq_4b$$

Hence 0011 is accepted by M.

$$q_0010 \vdash xq_110 \vdash q_2xy0 \vdash xq_0y0 \vdash xyq_30$$

As $\delta(q_3, 0)$ is not defined, M halts. So 010 is not accepted by M.

Example-3: Design TM that accepts $\{1^n2^n3^n \mid n \geq 1\}$

Solution:

Before designing the required Turing machine M, let us evolve a procedure for processing the input string 112233. After processing, we require the ID to be of the form $bbbbbbq_7$. The processing is done by using five steps:

Step 1 q_1 is the initial state. The R/W head scans the leftmost 1, replaces 1 by b, and moves to the right. M enters q_2.

Step 2 On scanning the leftmost 2, the R/W head replaces 2 by b and moves to the right. M enters q_3.

Step 3 On scanning the leftmost 3, the R/W head replaces 3 by b, and moves to the right. M enters q_4.

Step 4 After scanning the rightmost 3, the R/W heads moves to the left until it finds the leftmost 1. As a result, the leftmost 1, 2 and 3 are replaced by b.

Step 5 Steps 1–4 are repeated until all 1's, 2's and 3's are replaced by blanks.

Present state	*Input tape symbol*			
	1	2	3	b
$\rightarrow q_1$	bRq_2			bRq_1
q_2	$1Rq_2$	bRq_3		bRq_2
q_3		$2Rq_3$	bRq_4	bRq_3
q_4			$3Lq_5$	bLq_7
q_5	$1Lq_6$	$2Lq_5$		bLq_5
q_6	$1Lq_6$			bRq_1
(q_7)				

Transition diagram

The change of IDs due to processing of 112233 is given as

$$q_1112233 \vdash bq_212233 \vdash blq_22233 \vdash b1bq_3233 \vdash b1b2q_333$$

$$\vdash b1b2bq_43 \vdash b1b_2q_5b3 \vdash b1bq_52b3 \vdash b1q_5b2b3 \vdash bq_51b2b3$$

$$\vdash q_6b1b2b3 \vdash bq_11b2b3 \vdash bbq_2b2b3 \vdash bbbq_22b3$$

$$\vdash bbbbq_3b3 \vdash bbbbbq_33 \vdash bbbbbbq_4b \vdash bbbbbq_7bb$$

Thus,

$$q_1112233 \vdash^* q_7bbbbbb$$

It can be seen from the table that strings other than those of the form $0^n1^n2^n$ are not accepted.

Exercise: Compute the computation sequence for strings like 1223, 1123, 1233 and then see that these strings are rejected by *M*.

8.5 Description of Turing Machines

In the examples discussed so far, the transition function δ was described as a partial function δ: Q x Γ→ Q x Γ x {L, R} is not defined for all (q, x) by spelling out the current state, the input symbol, the resulting state, the tape symbol replacing the input symbol and the movement of R/W head to the left or right. We can call this a formal description of a TM.

Just as we have the machine language and higher-level languages for a computer, we can have a higher level of description, called the *implementation description.* In this case we describe the movement of the head, the symbol stored etc. in English. For example, a single instruction like "move to right till the end of the input string" requires several moves. A single instruction in the implementation description is equivalent to several moves of a standard TM. At a higher level we can give instructions in English language even without specifying the state or transition function. This is called a *high-level description.*

In the remaining sections of this chapter and later chapters, we give implementation description or high-level description.

8.6 Techniques for TM Construction

In this section we give some high-level conceptual tools to make the construction of TMs easier. The Turing machine defined earlier is called the standard Turing machine.

8.6.1. Turing Machine with Stationary Head

In the definition of a TM we defined $\delta(q, a)$ as (q', y, D) where $D = L$ or R. So the head moves to the left or right after reading an input symbol. Suppose, we want to include the option that the head can continue to be in the same cell for some input symbol. Then we define $\delta(q, a)$ as (q', y, S). This means that the TM, on reading the input symbol a, changes the state to q' and writes y in the current cell in place of a and continues to remain in the same cell. In terms of IDs,

$$wqax \vdash wq'yx$$

Of course, this move can be simulated by the standard TM with two moves, namely

$$wqax \vdash wyq''x \vdash wq'yx$$

That is, $\delta(q, a) = (q', y, S)$ is replaced by $\delta(q, a) = (q'', y, R)$ and $\delta(q'', X) = (q', y, L)$ for any tape symbol X.

Thus in this model $\delta(q, a) = (q', y, D)$ where $D = L, R$ or S.

8.6.2. Storage in the State

We are using a state, whether it is of a FSM or PDA or TM, to 'remember' things. We can use a state to store a symbol as well. So, the state becomes a pair *(q, a)* where q is the state (in the usual sense) and a is the tape symbol stored in *(q, a)*. So, the new set of states becomes Q x Γ.

Example: Construct a TM that accepts the language 0 1* + 1 0*.

Solution

We have to construct a TM that remembers the first symbol and checks that it does not appear afterwards in the input string. So we require two states, q_0, q_1. The tape symbols are 0, 1 and b. So the TM, having the 'storage facility in state', is

$$M = (\{q_0, q_1\} \times \{0, 1, b\}, \{0, 1\}, \{0, 1, b\}, \delta, [q_0, b], \{[q_1, b]\})$$

We describe δ by its implementation description.

1. In the initial state, M is in q_0 and has b in its data portion. On seeing the first symbol of the input sting w, M moves right, enters the state q_1 and the first symbol, say a, it has seen.
2. M is now in $[q_1, a]$. (i) If its next symbol is b, M enters $[q_1, b]$, an accepting state. (ii) If the next symbol is a, M halts without reaching the final state (i.e. δ is not defined). (iii) If the next symbol is $\overline{a}$ ($\overline{a} = 0$ if $a = 1$ and $\overline{a} = 1$ if $a = 0$), M moves right without changing state.
3. Step 2 is repeated until M reaches $[q_1, b]$ or halts (δ is not defined for an input symbol in w).

8.6.3. Multiple Track Turing Machine

In the case of TM defined earlier, a single tape was used. In a multiple track TM. a single tape is assumed to be divided into several tracks. Now the tape alphabet is required to consist of k-

tuples of tape symbols, k being the number of tracks. Hence the only difference between the standard TM and the TM with multiple tracks is the set of tape symbols. In the case of the standard Turing machine, tape symbols are elements of Γ; in the case of TM with multiple track, it is Γ^k. The moves are defined in a similar way.

8.6.4. Subroutines

We know that subroutines are used in computer languages, when some task has to be done repeatedly. We can implement this facility for TMs as well.

First a TM program for the subroutine is written. This will have an initial state and a 'return' state. After reaching the return state. there is a temporary halt. For using a subroutine, new states are introduced. When there is a need for calling the subroutine, moves are affected to enter the initial state for the subroutine (when the return state of the subroutine is reached) and to return to the main program of TM.

We use this concept to design a TM for performing multiplication of two positive integers.

Example: Design a TM which can multiply two positive integers.

Solution

The input (m, n), m, n being given, the positive integers are represented by 0^m10^n. M starts with 0^m10^n in its tape. At the end of the computation, 0^{mn}(mn in unary representation) surrounded by b's is obtained as the ouput.

The major steps in the construction are as follows:

1. 0^m10^n1 is placed on the tape (the output will be written after the rightmost 1).
2. The leftmost 0 is erased.
3. A block of n 0's is copied onto the right end.
4. Steps 2 and 3 are repeated m times and 10^m10^{mn} is obtained on the tape.
5. The prefix 10^m1 of 10^m10^{mn} is erased. leaving the product mn as the output.

For every 0 in 0^m, 0^n is added onto the right end. This requires repetition of step 3. We define a subroutine called COPY for step 3.

For the subroutine COPY. the initial state is q_1 and the final state is q_5. δ is given by the transition table (see Table 9.7).

TABLE 9.7 Transition Table for Subroutine COPY

State	Tape symbol			
	0	1	2	b
q_1	$q_2 2R$	$q_4 1L$	—	—
q_2	$q_2 0R$	$q_2 1R$	—	$q_3 0L$
q_3	$q_3 0L$	$q_3 1L$	$q_1 2R$	—
q_4	—	$q_5 1R$	$q_4 0L$	—
q_5	—	—	—	—

The Turing machine M has the initial state q_0. The initial ID for M is $q_0 0^m 10^n 1$. On seeing 0, the following moves take place (q_6 is a state of M). $q_0 0^m 10^n 1 \vdash bq_6 0^{m-1} 10^n 1 \vdash^* b0^{m-1} q_6 10^n 1 \vdash b0^{m-1} 1q_1 0^n 1$. q_1 is the initial state of COPY. The TM M_1 performs the subroutine COPY. The following moves take place for M_1: $q_1 0^n 1 \vdash 2q_2 0^{n-1} 1 \vdash^* 20^{n-1} 1q_3 b \vdash 20^{n-1} q_3 10 \vdash^* 2q_1 0^{n-1} 10$. After exhausting 0's, q_1 encounters 1. M_1 moves to state q_4. All 2's are converted back to 0's and M_1 halts in q_5. The TM M picks up the computation by starting from q_5. The q_0 and q_6 are the states of M. Additional states are created to check whether each 0 in 0^m gives rise to 0^m at the end of the rightmost 1 in the input string. Once this is over, M erases $10^n 1$ and finds 0^{mn} in the input tape.

M can be defined by

$$M = (\{q_0, q_1, \ldots, q_{12}\}, \{0, 1\}, \{0, 1, 2, b\}, \delta, q_0, b, \{q_{12}\})$$

where δ is defined by Table 9.8.

TABLE 9.8 Transition Table for Example 9.10

	0	1	2	b
q_0	q_8bR	—	—	—
q_6	q_60R	q_11R	—	—
q_5	q_70L	—	—	—
q_7	—	q_81L	—	—
q_8	q_90L	—	—	$q_{10}bR$
q_9	q_90L	—	—	q_0bR
q_{10}	—	$q_{11}bR$	—	—
q_{11}	$q_{11}bR$	$q_{12}bR$	—	—

Transition Diagram

Thus, M performs multiplication of two numbers in unary representation.