Regular and Non Regular Languages

2

Regular Languages

- There is a countably infinite number of regular languages.

  {a}, {aa} , {aaa}, {aaaa}. {aaaaa}, {aaaaaa }…

- Every finite language is regular.

  $s_1 \cup s_2 \cup \cdots \cup s_n$

3

Closure Properties of Regular Languages

- **Theorem: The regular languages are closed under union, concatenation, and Kleene star.**
- Proof: By the same constructions that were used in the proof of Kleene's theorem

4

Closure Properties of Regular Languages

- Theorem: The regular languages are closed under complement, intersection, difference, reverse, and letter substitution.

5

Closure Properties of Regular Languages

- Theorem: The regular languages are closed under **complement**, intersection, difference, reverse, and letter substitution.

- The regular languages are closed under complement. If L_1 is regular, then there exists a DFSM $M_1 = (K, \Sigma, \delta, s, A)$ that accepts it. The DFSM $M_2 = (K, \Sigma, \delta, s, K - A)$, namely M_1 with accepting and nonaccepting states swapped, accepts $\neg(L(M_1))$ because it rejects all strings that M_1 accepts and rejects all strings that M_1 accepts.

6

Closure Properties of Regular Languages

- Theorem: The regular languages are closed under complement, **intersection**, difference, reverse, and letter substitution.

- The regular languages are closed under intersection. We note that:

$$L(M_1) \cap L(M_2) = \neg(\neg L(M_1) \cup \neg L(M_2)).$$

We have already shown that the regular languages are closed under both complement and union. Thus they are also closed under intersection.

7

Closure Properties of Regular Languages

- Theorem: The regular languages are closed under complement, intersection, **difference**, reverse, and letter substitution.

- The regular languages are closed under set difference (subtraction). We note that:

$$L(M_1) - L(M_2) = L(M_1) \cap \neg L(M_2).$$

We have already shown that the regular languages are closed under both complement and intersection. Thus they are also closed under set difference.

8

Closure Properties of Regular Languages

- Theorem: The regular languages are closed under complement, intersection, difference, reverse, and letter substitution.

- The regular languages are closed under reverse. Recall that $L^R = \{w \in \Sigma^* : w = x^R \text{ for some } x \in L\}$. We leave the proof of this as an exercise.
- The regular languages are closed under letter substitution, defined as follows: Consider any two alphabets, Σ_1 and Σ_2. Let *sub* be any function from Σ_1 to Σ_2^*. Then *letsub* is a letter substitution function from L_1 to L_2 iff $letsub(L_1) = \{w \in \Sigma_2^* : \exists y \in L_1 (w = y$ except that every character c of y has been replaced by $sub(c))\}$. For example, suppose that $\Sigma_1 = \{a, b\}$, $\Sigma_2 = \{0, 1\}$, $sub(a) = 0$, and $sub(b) = 11$. Then $letsub(\{a^n b^n : n \geq 0\}) = \{0^n 1^{2n} : n \geq 0\}$. We leave the proof that the regular languages are closed under letter substitution as an exercise.

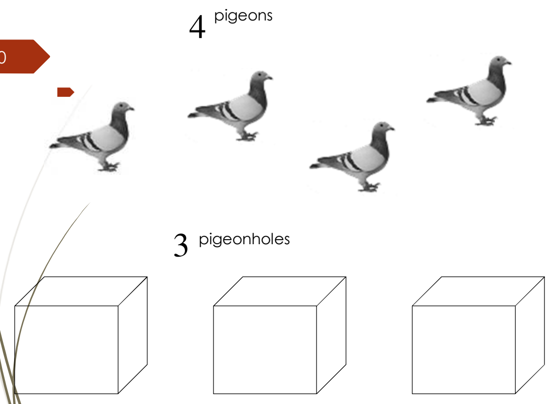
9

Long Strings Force Repeated States

Theorem: Let $M = (K, \Sigma, \delta, s, A)$ be any DFSM. If M accepts any string of length $|K|$ or greater, then that string will force M to visit some state more than once (thus traversing at least one loop).

Proof: M must start in one of its states. Each time it reads an input character, it visits some state. So, in processing a string of length n, M creates a total of $n + 1$ state visits (the initial one plus one for each character it reads). If $n + 1 > |K|$, then, by the pigeonhole principle, some state must get more than one visit. So, if $n \geq |K|$, then M must visit at least one state more than once.

10



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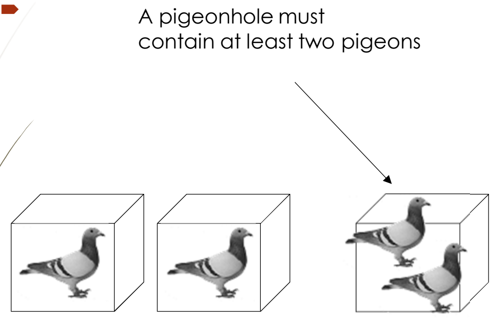
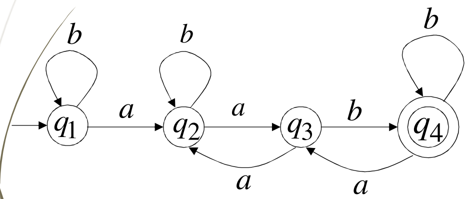


12

Consider a DFA with 4 states

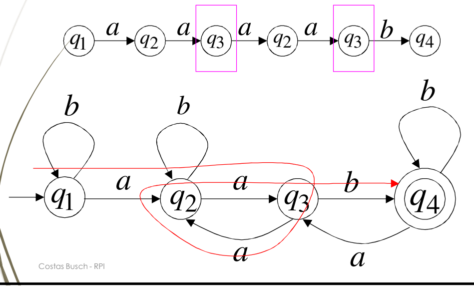


13

Consider the walk of a "long" string: $aaaab$

(length at least 4)

A state is repeated in the walk of $aaaab$



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14

Showing That a Language is Not Regular

The Pumping Theorem for Regular languages

Theorem: If L is a regular language, then:

$\exists k \geq 1$ ($\forall$ strings $w \in L$, where $|w| \geq k$ $(\exists x, y, z$ $(w = xyz,$
$|xy| \leq k,$
$y \neq \varepsilon$, and
$\forall q \geq 0$ $(xy^q z \in L)))).$

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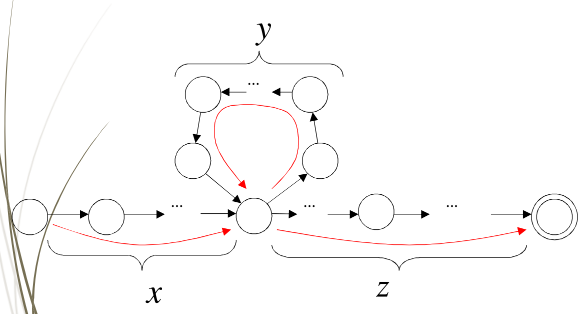
- If L is regular then it is accepted by some DFSM $M = (K, \Sigma, \delta, s, A)$.
- Let k be IKI.
- Let w be any string in L of length k or greater.
- By previous theorem to accept w, M must traverse some loop at least once.
- We can carve w up and assign the name y to the first substring to drive M through a loop.
- Then x is the part of w that precedes y and z is the part of w that follows y .
- **We show that each of the last three conditions must then hold:**

16

In DFSM: $w = x\ y\ z$



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- We show that each of the last three conditions must then hold:

- $|xy| \leq k$: M must not only traverse a loop eventually when reading w, it must do so for the first time by at least the time it has read k characters. It can read $k - 1$ characters without revisiting any states. But the k^{th} character must, if no earlier character already has, take M to a state it has visited before. Whatever character does that is the last in one pass through some loop.
- $y \neq \varepsilon$: Since M is deterministic, there are no loops that can be traversed by ε.

Theorem: If L is a regular language, then:

$\exists k \geq 1$ ($\forall$ strings $w \in L$, where $|w| \geq k$ $(\exists x, y, z$ $(w = xyz,$
$|xy| \leq k,$
$y \neq \varepsilon$, and
$\forall q \geq 0$ $(xy^q z \in L)))).$

- We show that each of the last three conditions must then hold:

- $\forall q \geq 0$ $(xy^q z \in L)$: y can be pumped out once (which is what happens if $q = 0$) or in any number of times (which happens if q is greater than 1) and the resulting string must be in L since it will be accepted by M. It is possible that we could chop y out more than once and still generate a string in L, but without knowing how much longer w is than k, we don't know any more than that it can be pumped out once.

19

Applications of Pumping Lemma

- Pumping Lemma is to be applied to show that certain languages are not regular.
- It should never be used to show a language is regular.
  - If **L** is regular, it satisfies Pumping Lemma.
  - If **L** does not satisfy Pumping Lemma, it is non-regular.

20

- The Pumping Theorem is a powerful tool for showing that a language is not regular.
- But, as with any tool, using it effectively requires some skill, To see how the theorem can be used,
- We can state it, in most general terms:

  For any language *L*, if *L* is regular, then every "long" string in *L* is pumpable.

In a nutshell then, to use the Pumping Theorem to show that a language L is not regular, we must:

1. Choose a string w, where $w \in L$ and $|w| \geq k$. Note that we do not know what k is; we know only that it exists. So we must state w in terms of k.
2. Divide the possibilities for y into a set of equivalence classes so that all strings in a class can be considered together.
3. For each such class of possible y values, where $|xy| \leq k$ and $y \neq \varepsilon$:
   Choose a value for q such that xy^qz is not in L.

22

Prove that **L = {a^nb^n | n ≥ 0}** is not regular.

- If **L** is regular, ∃k ∀w∈L |w| ≥ k
- Let $w = a^kb^k$. Thus |w| = 2k ≥ k.
- By pumping lemma, let w = xyz, where |xy| ≤ k.
- To guarantee |xy| ≤ k, y must occur with first k characters, so $y=a^p$, for some p
- To guarantee y≠ ε, ⇒ p>0 and the resulting string is $a^{n+q}b^n$,
- As per the last condition of the pumping lemma, the string $a^{n+q}b^n$ must be in L. But in reality it is not
- Thus, xy^2z is not in L. Hence L is not regular.

23

Prove that Balanced Parenthesis Language is Not Regular

Let L be Bal = {$w \in$ {), (}* : the parentheses are balanced}. If L were regular, then there would exist some k such that any string w, where $|w| \geq k$, must satisfy the conditions of the theorem. Bal contains complex strings like (())(()()). But it is almost always easier to use the Pumping Theorem if we pick as simple a string as possible. So, let $w = (^k)^k$. Since $|w| = 2k$ and w is in L, w must satisfy the conditions of the Pumping Theorem. So there must exist x, y, and z, such that $w = xyz$, $|xy| \leq k$, $y \neq \varepsilon$, and $\forall q \geq 0\ (xy^qz \in L)$. But we show that no x, y, and z exist. Since $|xy| \leq k$, y must occur within the first k characters and so $y = (^p$ for some p). Since $y \neq \varepsilon$, p must be greater than 0. Let $q = 2$. (In other words, we pump in one extra copy of y.) The resulting string is $(^{k+p})^k$. The last condition of the Pumping Theorem states that this string must be in L, but it is not since it has more ('s than)'s. There exists at least one long string in L that fails to satisfy the conditions of the Pumping Theorem. So L = Bal is not regular.

24

Prove Even Palindrome language is Not Regular

Let L be PalEven = { $xx^R : x \in \{a,b\}^*$ }. PalEven is the language of even-length palindromes of a's and b's. We can use the Pumping Theorem to show that PalEven is not regular. If it were, then there would exist some k such that any string w, where $|w| \geq k$, must satisfy the conditions of the theorem. We show one string w that does not. (Note here that the variable w used in the definition of L is different from the variable w mentioned in the Pumping Theorem.) We will choose w so that we only have to consider one case for where y could fall. Let $w = a^kb^kb^ka^k$. Since $|w| = 4k$ and w is in L, w must satisfy the conditions of the Pumping Theorem. So there must exist x, y, and z, such that $w = xyz$, $|xy| \leq k$, $y \neq \varepsilon$, and $\forall q \geq 0\ (xy^qz \in L)$. Since $|xy| \leq k$, y must occur within the first k characters and so $y = a^p$ for some p. Since $y \neq \varepsilon$, p must be greater than 0. Let $q = 2$. The resulting string is $a^{k+p}b^kb^ka^k$. If p is odd, then this string is not in PalEven because all strings in PalEven have even length. If p is even then it is at least 2, so the first half of the string has more a's than the second half does, so it is not in PalEven. So L = PalEven is not regular.

25

The Language with More a's Than b's is Not Regular

Let $L = \{a^n b^m : n > m\}$. We can use the Pumping Theorem to show that L is not regular. If it were, then there would exist some k such that any string w, where $|w| \geq k$, must satisfy the conditions of the theorem. We show one string w that does not. Let $w = a^{k+1}b^k$. Since $|w| = 2k + 1$ and w is in L, w must satisfy the conditions of the Pumping Theorem. So there must exist x, y, and z, such that $w = xyz$, $|xy| \leq k$, $y \neq \varepsilon$, and $\forall q \geq 0\ (xy^qz \in L)$. Since $|xy| \leq k$, y must occur within the first k characters and so $y = a^p$ for some p. Since $y \neq \varepsilon$, p must be greater than 0. There are already more a's than b's, as required by the definition of L. If we pump in, there will be even more a's and the resulting string will still be in L. But we can set q to 0 (and so pump out). The resulting string is then $a^{k+1-p}b^k$. Since $p > 0$, $k + 1 - p \leq k$, so the resulting string no longer has more a's than b's and so is not in L. There exists at least one long string in L that fails to satisfy the conditions of the Pumping Theorem. So L is not regular.

26

The Prime Number of a's Language is Not Regular

Let L be $\text{Prime}_a = \{a^n : n \text{ is prime}\}$. We can use the Pumping Theorem to show that L is not regular. If it were, then there would exist some k such that any string w, where $|w| \geq k$, must satisfy the conditions of the theorem. We show one string w that does not. Let $w = a^j$, where j is the smallest prime number greater than $k + 1$. Since $|w| > k$, w must satisfy the conditions of the Pumping Theorem. So there must exist x, y, and z, such that $w = xyz$, $|xy| \leq k$ and $y \neq \varepsilon$. $y = a^p$ for some p. The Pumping Theorem further requires that $\forall q \geq 0\ (xy^qz \in L)$. So, $\forall q \geq 0\ (a^{|x|+|z|+q \cdot |y|}$ must be in L). That means that $|x| + |z| + q \cdot |y|$ must be prime.

But suppose that $q = |x| + |z|$. Then:

$$\begin{aligned} |x| + |z| + q \cdot |y| &= |x| + |z| + (|x| + |z|) \cdot y \\ &= (|x| + |z|) \cdot (1 + |y|), \end{aligned}$$

which is composite (non-prime) if both factors are greater than 1. $(|x| + |z|) > 1$ because $|w| > k + 1$, and $|y| \leq k$. $(1 + |y|) > 1$ because $|y| > 0$. So, for at least that one value of q, the resulting string is not in L. So L is not regular.

27

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Thank you

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